

A New Paradigm for the Fermion Generations

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Abstract

A new mechanism is proposed to explain the appearance of the three known fermion generations in a natural way. The underlying idea is based on the discreteness of the spectrum of solutions of the gap equation appearing in models of dynamical chiral symmetry breaking. Within such a framework, the number of parameters needed to describe the experimentally observed fermion spectrum is drastically reduced. The phenomenological consequences of such a mechanism are carefully discussed, in order to explore its viability.

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1 Introduction

After many years of theoretical investigations focused on possible mechanisms responsible for the electroweak symmetry breaking and the generation of fermion masses, the puzzle still remains unsolved. Even though the minimal Higgs mechanism seems to be consistent with current experimental data [1], it remains conceptually unsatisfactory, due to the excessive fine tuning that must be applied to the Higgs coupling in order to keep the renormalized Higgs mass reasonably close to the weak scale.

The two presently most prominent alternatives to the Higgs mechanism are supersymmetry and technicolor, even though they could be far from providing the correct mechanism hidden behind the Higgs sector. They tackle the problem of weak $SU(2)_L$ breaking in quite different ways. Nevertheless, they have a common feature: they associate each fermion mass with a different coupling, a Higgs coupling when it comes to supersymmetry, and an extended-technicolor coupling when it comes to technicolor. This loads these two theories with too many parameters, making them unnatural in that respect. Model-builders for both theories typically hide this problem “under the rug”, placing the natural origins of these couplings, and their effects to the fermion family replication, to unknown physics at much higher energies.

There have been attempts to reduce the number of free parameters by predicting the number of fermion generations within the framework of grand-unified theories (supersymmetric or not) [2], compositeness models [3], string theory [4], as well as many other ideas [5]. In most of these cases, however, what is naturally derived is the number of generations, and not the particular scale of the fermion masses in these generations. Moreover, in the case of GUT, the number of generations is presented as a constraint imposed by

phenomenology, with no fundamental explanation. In supersymmetry, there has been a recent attempt to reduce considerably the number of free parameters entering the fermion spectrum of the theory [6]. Although such an attempt seems to be headed towards the correct direction, it is still plagued with a draw-back: the number of fermion generations, and the hierarchy between them, is again introduced *ad hoc*, with no underlying mechanism presented as responsible for it.

Moreover, recent extended-technicolor models, in their attempt to decrease the technicolor contribution to the $\Delta\rho$ parameter, introduce even more parameters, associating with each ordinary fermion not only each own extended-technicolor coupling, but also each own extended-technicolor scale [7]. The idea of multiple scales appeared in the early days of technicolor theories, in the context of “tumbling” [8], which still remains a popular idea [9]. In all these studies, however, the hierarchy of the fermion generation scales is put in by hand, without being presented within a theoretical framework that would justify their magnitude or multiplicity. General model building considerations might give one an idea of what these scales should be, but it is our feeling that they do not tackle the problem in the most fundamental way.

Such a situation is obviously far from desirable for a natural theory. Especially for technicolor, since one introduces additional degrees of freedom, in the form of a new non-abelian gauge group and families of new, presently unobserved, fermions, one would expect a much smaller number of parameters needed to explain the ordinary fermion spectrum, without having to resort to fermion compositeness or “barock” models.

The present paper proposes a mechanism that could potentially explain how the known fermions acquire their masses and they are at the same time placed in distinct generations. The analysis is done within the framework of technicolor theories, since it is

not presently clear to the author how a mechanism based on similar principles could be at works in supersymmetric theories. The method relies heavily on the Schwinger-Dyson gap equations, solved by using the typical assumptions and approximations that are used in technicolor models [10]. The idea central to the development of the paper is that the new physics, introduced by the extended-technicolor interactions at a scale Λ_{ETC} , the scale at which the extended-technicolor group breaks, act as an effective cut-off to the integral gap equations which give the self-energies of the fermions. This provides the theory with a discrete spectrum of solutions.

An attempt is then made to associate the first three solutions of this discrete spectrum with the three fermion generations, and to find physical arguments that would allow the truncation of this spectrum beyond the third solution, since present electroweak-experiment data [1] constrain the fermion generations to three. The method gives results that are directly testable, since it produces explicit values for the order of magnitude of masses and scales that can be easily discarded if they grossly contradict phenomenology. At first sight, it is not apparent to us how such a mechanism could tell us something precise about the CKM matrix, so we are not going to address that problem here. Moreover, it should be noted that, because the equations that appear in this analysis are very difficult to solve exactly, the results presented here try to sketch the qualitative features of the mechanism, with no ambition for providing quantitatively reliable results.

This is how this work is organized: At first, the general setting of technicolor theories is overviewed, including extended-technicolor interactions. Then, the role that the integral Schwinger-Dyson equations play in the dynamical symmetry breaking is analysed in a somewhat novel way, and it is made clear how their properties could help us solve the fermion-spectrum puzzle. The results of this analysis show that a certain combination of

the physical parameters of some models obey a quantization condition. The next section tries to motivate such a quantization condition physically in various ways, and to check whether the predictions of this mechanism are consistent with current phenomenology. The final section summarizes the conclusions drawn by this analysis, and attempts to test the naturalness and viability of the proposed mechanism.

2 Mass generation in technicolor

As was mentioned in the previous section, the Higgs mechanism, in trying to explain the breaking of the $SU(2)_L$ gauge group and the origin of fermion masses, seems to describe these phenomena correctly, but it has a naturalness problem, since too much fine tuning of the Higgs coupling is required in order to keep the renormalized Higgs mass acceptably small. One of the alternatives proposed in order to circumvent this problem is provided by supersymmetry, a “weak coupling” alternative, which introduces additional Higgs fields, but at the same time solves the naturalness problem. The other one is technicolor, a “strong coupling” alternative, which postulates the existence of new fermions, called technifermions, which interact strongly with each other via a technicolor gauge interaction. In such a framework, the role of the Higgs fields is played by condensates of technifermion pairs. The subject of the present paper is centered on this second alternative.

Technicolor theory is based on the *ad hoc* introduction of N_f new fermions, initially massless, not experimentally detected yet, and having a new quantum number called technicolor [10]. The gauge group responsible for their mutual interactions, traditionally called technicolor group, leads to confinement and to the dynamical break-down of the initial global chiral $SU(N_f)_L \times SU(N_f)_R$ symmetry down to $SU(N_f)_V$. Due to Goldstone’s theorem, this leads to the appearance of $N_f^2 - 1$ massless Goldstone bosons. In

such a scenario, three of them are “eaten” by the electroweak gauge bosons W^\pm, Z^0 , and the rest $N_f^2 - 4$ become pseudo-Goldstone bosons (PGBs), after acquiring masses due to the explicit chiral symmetry breaking by the conventional Standard-Model interactions $SU(3) \times SU(2)_L \times U(1)$. These PGBs are composite particles, consisting of 2 technifermions, and they are singlets under the technicolor group. If this mechanism causes the breaking of the electroweak symmetry, the order of magnitude of the scale Λ_{TC} of the technicolor group, where confinement of the technifermions occurs, should be on the order of 1 TeV.

Even though the above mechanism can explain the masses of the gauge bosons of weak interactions, it does not explain the masses of ordinary fermions. This problem is solved by postulating the existence of a new interaction, called extended technicolor (ETC), that is associated with a gauge group that is broken at an energy scale Λ_{ETC} , usually much larger than Λ_{TC} . Both ordinary fermions and technifermions feel this interaction, which, at scales close to 1 TeV, manifests itself in the form of effective (non-renormalizable) 4-fermion interactions among fermions and technifermions. Thus, a condensate of two technifermions (T) can “feed down” its mass to ordinary fermions (f), via an interaction of the form

$$\frac{\lambda_{ETC}^2}{\Lambda_{ETC}^2} \bar{f}_L T_L \bar{T}_R f_R, \quad (1)$$

where λ_{ETC} is the effective ETC coupling. The fermion masses are then given by

$$m_f \approx \frac{\lambda_{ETC}^2}{\Lambda_{ETC}^2} \langle \bar{T} T \rangle. \quad (2)$$

However, one should also expect effective ETC interactions of the form

$$\frac{1}{\Lambda_{ETC}^2} \bar{f}_L f_L \bar{f}_R f_R, \quad (3)$$

which could potentially lead to problems with too large flavor-changing neutral currents (FCNC). In order to avoid that, the ETC scale Λ_{ETC} must be taken very large, on the order of about 1000 TeV. This leads to very small fermion masses, according to Eq.2, and it certainly cannot account for the masses of the heavier quarks, unless an excessive fine tuning of the effective ETC coupling is used. This would unfortunately lead us back to the naturalness problem, a problem that technicolor was created in order to avoid. A solution to this problem was proposed some years ago [11], in the form of “walking” technicolor models, in which the technicolor gauge coupling runs slowly due to the screening of the technicolor charge by the technifermions. This mechanism allows for large fermion masses, while adequately suppressing FCNC.

During the last decade, the testing of the phenomenological consequences of technicolor models made it very important to study very carefully the momentum dependence of the self-energy of the technifermions, i.e. the transition from the “constituent” technifermion masses, at low energies, to the “current” technifermion masses, at high energies. A way of performing such a study is provided by the CJT formalism [12], which was originally developed in the QCD context, but can in principle be applied to any strongly interacting theory. In this formalism, one starts from a Lagrangian describing the strongly interacting fermions, and after using effective action techniques one writes down a Schwinger-Dyson equation for the self-energies of these fermions. The analysis here below follows closely Ref.[10].

For a Lagrangian density of the form

$$L = \bar{\psi}\gamma_\mu(i\partial^\mu - gA^\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (4)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ is the curvature of the technicolor gauge group, and by ψ we denote the technifermion fields, this formalism gives us the following Schwinger-Dyson equation, in the ladder approximation, and neglecting the running of the gauge coupling g :

$$S^{-1} = p^\mu \gamma_\mu + ig^2 C_2(R) \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k) \gamma_\nu \frac{g^{\mu\nu} - (1 - \xi) \frac{(p-k)^\mu (p-k)^\nu}{(p-k)^2}}{(p-k)^2}, \quad (5)$$

where S is the fermion propagator, ξ allows for different gauge choices, and $C_2(R)$ is the quadratic Casimir invariant of the fundamental representation of the technicolor group. For $SU(N_{TC})$, $C_2(R) = \frac{N_{TC}^2 - 1}{2N_{TC}}$.

By making now the ansatz $S^{-1} = A(p^2)\gamma^\mu p_\mu - \Sigma(p^2)$, where $\Sigma(p^2)$ is the fermion self-energy, and in Euclidean space, after angular integration, we get a set of equations:

$$\begin{aligned} A(p^2) &= 1 + \frac{\xi a}{3} \int_0^{\Lambda^2} dk^2 \frac{k^4}{M^4} \frac{A(k^2)}{A^2(k^2)k^2 + \Sigma^2(k^2)} \\ \Sigma(p^2) &= \frac{(3 + \xi)a}{3} \int_0^{\Lambda^2} dk^2 \frac{k^2}{M^2} \frac{\Sigma(k^2)}{A^2(k^2)k^2 + \Sigma^2(k^2)}, \end{aligned} \quad (6)$$

where $M = \max(p, k)$, $a = \frac{\alpha}{4\alpha_c}$, with $\alpha = g^2/4\pi$ and $\alpha_c = \pi/3C_2(R)$. We have placed a UV cut-off to our theory. In technicolor theories, Λ is the typical ETC scale. In the Landau gauge, where $\xi = 0$, we have $A = 1$. If we want to have a gauge coupling strong enough to break dynamically the chiral symmetry of the theory, and at the same time small enough to justify the use of perturbation theory in the CJT formalism, the relation $1/4 < a < 1$ must hold.

In what is called the “dressed” ladder approximation, we have a more physical situation, by allowing the gauge coupling to run, so that $\alpha = \alpha((p - k)^2)$. In a certain approximation then, we get the same equations as above, but now the coupling a appears inside the integrals over k .

Now, for an effective, non-renormalizable Lagrangian that contains 4-fermi interactions, like ETC interactions, of the form

$$L = \bar{\psi} i \partial_\mu \gamma^\mu \psi + \frac{g}{2} (\bar{\psi} \gamma_\mu \psi) (\bar{\psi} \gamma^\mu \psi), \quad (7)$$

the CJT formalism gives an inverse fermion propagator of the form

$$S^{-1} = p^\mu \gamma_\mu + ig \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k) \gamma^\mu, \quad (8)$$

so we can write $S^{-1}(p^2) = p^\mu \gamma_\mu - \Sigma$, where Σ is independent of p . We then get an equation for the fermion self-energy Σ :

$$\Sigma = \lambda \int_0^{\Lambda^2} dk^2 \frac{k^2}{\Lambda^2} \frac{\Sigma}{k^2 + \Sigma^2}, \quad (9)$$

with $\lambda = \frac{g\Lambda^2}{4\pi^2}$, and Λ is a UV cut-off in the theory, necessary in order to avoid the logarithmic divergence of Σ .

Combining the results obtained above, we can study a technicolor theory that contains both 4-fermi and gauge interactions. In the Landau gauge and in Euclidean space, we get

$$\begin{aligned}
\Sigma(p^2) &= \int_0^{\Lambda^2} dk^2 a(M^2) \frac{k^2}{M^2} \frac{\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} + \\
&+ \lambda_{TT} \int_0^{\Lambda^2} dk^2 \frac{k^2}{M^2} \frac{\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} + \lambda_{Tf} \int_0^{\Lambda^2} dk^2 \frac{k^2}{\Lambda^2} \frac{m_f}{k^2 + m_f^2} \\
m_f &= \lambda_{ff} \int_0^{\Lambda^2} dk^2 \frac{k^2}{\Lambda^2} \frac{m_f}{k^2 + m_f^2} + \lambda_{fT} \int_0^{\Lambda^2} dk^2 \frac{k^2}{M^2} \frac{\Sigma(k^2)}{k^2 + \Sigma^2(k^2)}, \tag{10}
\end{aligned}$$

where $\Sigma(p^2)$ is the the technifermion self-energy, and m_f is the mass of an ordinary fermion coupled to the technifermion via 4-fermi ETC interactions. $\Sigma(p^2)$ receives contributions from the technicolor gauge interactions, as well as from ETC 4-fermi interactions of the technifermion with other technifermions (see term proportional to λ_{TT}), and with ordinary fermions (see term proportional to λ_{Tf}). The mass of the fermion receives contributions only by 4-fermi interactions of the fermion with other ordinary fermions (see term proportional to λ_{ff}), and with technifermions (see term proportional to λ_{fT}).

After making the approximation

$$\int_0^{\Lambda^2} dk^2 \frac{k^2}{\Lambda^2} \frac{m_f}{k^2 + m_f^2} \approx m_f \left(1 - \frac{m_f^2}{\Lambda^2} \ln \left(\frac{\Lambda^2}{m_f^2} \right) \right) \approx m_f, \tag{11}$$

which is valid for scales Λ large compared to the fermion mass, we get the gap equations

$$\begin{aligned}
\Sigma(p^2) &= \int_0^{\Lambda^2} dk^2 a(M^2) \frac{k^2}{M^2} \frac{\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} + \lambda_T \int_0^{\Lambda^2} dk^2 \frac{k^2}{\Lambda^2} \frac{\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \\
m_f &= \lambda_f \int_0^{\Lambda^2} dk^2 \frac{k^2}{\Lambda^2} \frac{\Sigma(k^2)}{k^2 + \Sigma^2(k^2)}, \tag{12}
\end{aligned}$$

with $\lambda_f \equiv \frac{\lambda_{fT}}{1-\lambda_{ff}}$ and $\lambda_T \equiv \lambda_{TT} + \frac{\lambda_{Tf}\lambda_{fT}}{1-\lambda_{ff}}$. We note that the second term of the right-hand side of the equation giving $\Sigma(p^2)$ is momentum independent, so it enters in the problem through the boundary conditions of the differential equation corresponding to Eq.12. Having arrived at this result, we are ready to proceed in a careful analysis of the behavior of these integral equations. In the next section, we try to analyse analogous integral equations, which can be viewed as simplifications of the above relations, and which permit us to study their behavior in a simple way.

3 The role of integral equations in technicolor models

3.1 Analytical study

In the following, we try to describe the problem of dynamical mass generation in an order of growing sophistication. The simplest way that an integral gap equation enters in the problem of mass generation is through the Nambu-Jona-Lasinio model [13]. There, we are confronted with an equation of the form

$$m = \frac{a}{\Lambda^2} \int_0^{\Lambda^2} \frac{mp^2 dp^2}{p^2 + m^2}, \quad (13)$$

where m is the mass of the fermion, which is taken to be momentum-independent, Λ is the UV cut-off of the theory, and a is a coupling associated with 4-fermion interactions. This is essentially the same as Eq.9. Apart from the trivial solution $m = 0$, the above equation possesses a solution given by the equation

$$\frac{1}{\lambda} = 1 - \frac{m^2}{\Lambda^2} \ln \left(\frac{\Lambda^2}{m^2} + 1 \right). \quad (14)$$

Such an equation exhibits a critical behavior, since, for $\lambda < 1$, it does not possess a non-trivial solution.

If we want to account for momentum-dependent masses, things get more complicated. One has to resort to the Schwinger-Dyson-equations formalism, and the resulting equations are quite intractable. As we saw in the previous section, one usually has to make the ladder approximation and go to Euclidean space and to the Landau gauge in order to simplify the equations, which are still analytically solvable only in the high- and low-energy limits.

Even though all these manipulations make us view the final results with scepticism, it is possible that they describe the qualitative behavior of the theory correctly. However, since their precise form is questionable, we will try to analyse a different, but very similar, integral gap equation, that could be relevant to our problem ¹. Namely, we are going to follow the reverse procedure from the usual one, by making a particular ansatz for the functional form of the fermion self-energy, because we feel that it sheds light on some other aspects of dynamical mass generation. In that way, we will be able to concentrate on the quantization condition, instead of being lost in complicated gap equations, which are of questionable validity anyway.

First, we make an ansatz for the momentum-dependent fermion self-energy, $\Sigma(p^2)$, by assuming that, for $p \gg \Sigma_0$, where Σ_0 is the value of $\Sigma(p^2)$ near the chiral symmetry breaking scale, it takes the form $\Sigma(p^2) \approx \Sigma_0(p^2/\Sigma_0^2)^{-\gamma}$. We can then try to find what kind of integral equation a function like $\Sigma(p^2)$ satisfies, and then try to motivate it physically.

We first give two useful identities

$$\int_0^\infty e^{-st} t^{\gamma-1} dt = \Gamma(\gamma) s^{-\gamma} \quad (15)$$

¹ It is worth noting that a very similar analysis can be done by using the differential form of the gap equations, which yields the same results.

$$\int_0^\infty e^{-st} t^{-\gamma} dt = \Gamma(1-\gamma) s^{\gamma-1},$$

where Γ is the usual Γ -function, and $0 < \text{Re}(\gamma) < 1$. Combining the above equations, we get an eigenvalue integral equation of the form

$$G(s) = \tilde{\lambda} \int_0^\infty e^{-st} G(t) dt, \quad (16)$$

with eigenfunction $G(s) = \left(\sqrt{\Gamma(\gamma)} s^{-\gamma} - \sqrt{\Gamma(1-\gamma)} s^{\gamma-1} \right)$, and eigenvalue

$$\tilde{\lambda} = \left(\sqrt{\Gamma(\gamma)\Gamma(1-\gamma)} \right)^{-1} = \sqrt{\frac{\sin(\pi\gamma)}{\pi}}.$$

So, even though the functions $s^{-\gamma}$ and $s^{\gamma-1}$ are not separately eigenfunctions of the integral operator $\int_0^\infty e^{-st} [\dots] dt$, their specific linear combination given above is an eigenfunction. Unfortunately, it is not clear how one could interpret physically the integration measure $e^{-st} dt$ in Eq.16.

By applying the same integral operator twice, however, we end-up with the following equation:

$$G(s) = \lambda \int_0^\infty \frac{G(t)}{s+t} dt, \quad (17)$$

where $G(t)$ is the same as above, again with $0 < \text{Re}(\gamma) < 1$, and $\lambda = \frac{\sin(\pi\gamma)}{\pi}$. In purely mathematical terms, we are dealing now with a function $G(s)$ which, up to a numerical coefficient, which is the eigenvalue of the equation, is its own simple Stieltjes transform. This time, the integration measure $\frac{dt}{s+t}$ is much easier to interpret physically. Making the correspondence of $G(s)$ with the fermion self-energy, we can also make the correspondence of the s and t variables with squares of 4-momenta, say $s = p^2$ and $t = k^2$. Then, this integration measure is very close to the one appearing in the Schwinger-Dyson (S-D) equations in Eq.12.

More precisely, the integration measure before performing the angular integration that gives Eq.12 is of the form (see, for instance, Ref.[10]) $\frac{1}{\pi^2(p-k)^2} \frac{dk^4}{k^2 + \Sigma^2(k^2)}$. After angular integration, where spherical symmetry of the self-energy is assumed, the term $(p-k)^2$ is replaced by the quantity $M \equiv \max(p^2, k^2)$ (see Eq.12)². The integration measure appearing in Eq.17 is equivalent to approximating M with the quantity $p^2 + k^2$, and, in addition, neglecting the self-energy $\Sigma(k^2)$ appearing in the denominator. Therefore, Eq.17 is a linearized version of the usual S-D equations, and it is expected to give trustworthy results only in the limit $p^2 \gg \Sigma_0^2$. In that limit, our integration measure is a special case of a more general form that has appeared in the literature, giving similar results [14]. In such a context, one should not worry about the small-s behavior of the eigenfunction $G(s)$, which possesses a singularity in that region; a singularity which is integrable but nevertheless unphysical.

In all that discussion, we also neglect the extended technicolor contributions to the technifermion self-energy, since their effect can be usually absorbed in the UV boundary conditions.

An interesting feature of Eq. 17 is that, in general, the two terms $t^{-\gamma}$ and $t^{\gamma-1}$ are separately eigenfunctions of the same integral operator and the same eigenvalue. It is not necessary to take the specific linear combination used in Eq.16 any more. However, if we take the exponent γ to be complex, a situation that will appear in the next sections of this paper, and we insist on having a real eigenfunction, the situation changes: we then require γ to be of the form $\gamma = 1/2 + i\delta$, with δ a real number. In addition, we have to keep the linear combination that we have used so far, since $(\Gamma(\gamma))^* = \Gamma(1 - \gamma)$, or, for

²Even though this replacement is exact in the case of a non-running gauge coupling, it is just an approximation in the running case.

this matter, any linear combination of the two terms, where one of the coefficients is the complex conjugate of the other.

It is also interesting to note that the form of the eigenfunction $G(s)$ given above is not the most general one corresponding to the Stieltjes kernel in the special case where $\gamma = 1/2$. In that case, a more general form is $c_1 s^{-1/2} + c_2 s^{-1/2} \ln(s)$, where $c_{1,2}$ are arbitrary constants. The eigenvalue associated with this solution is exactly the same as the one for general γ . We are not going to occupy ourselves with this special case any further.

We can try to recover now the form of the solutions that the S-D equations give in Ref.[10]. In that analysis, the coupling λ satisfies the relation $\lambda = \frac{\alpha}{4\alpha_c}$, where the couplings are taken to be momentum independent for the moment. That means that, since in our case the coupling and the exponent γ of the eigenfunction are related by the equation $\lambda = \sin(\pi\gamma)/\pi = \cosh(\delta\pi)/\pi$, in the regime $\delta\pi \ll 1$ we get $\delta \approx \frac{2}{\pi} \sqrt{\lambda\pi - 1} = \frac{2}{\pi} \sqrt{\frac{\pi\alpha}{4\alpha_c} - 1}$. The quantity corresponding to δ in the exponent of the eigenfunctions used in Ref.[10] is equal to $\sqrt{\alpha/\alpha_c - 1}/2$, which is reasonably close to our expression, given the different integration measure used in the two cases, and the fact that we expanded our expression for the coupling for small δ , i.e. for $\lambda \approx 1/\pi$. Moreover, for the proposed form of the quantity γ , the above eigenvalue equation exhibits a critical behavior³, since, for real values of δ , it possesses non-trivial solutions only for $\lambda > 1/\pi \approx 0.32$.

Finally, by taking the factor multiplying one of the two solutions to be equal to $\Sigma_0^2 e^{i2\delta\theta}$, and taking care of the correct dimensionality of the quantities used, the resulting expression for the self-energy is

³ Note that $\cosh(x) > 1$, for all real numbers x .

$$\Sigma(p^2) = \frac{\Sigma_0^2}{p} \sin [2\delta(\ln(p/\Sigma_0) + \theta)], \quad (18)$$

a well-known functional form in the technicolor literature [10]. Here Σ_0 is the characteristic energy of the theory, which is on the order of the chiral symmetry breaking scale.

Unfortunately, such a solution in general possesses nodes, and momentum regions at which the self-energy becomes negative. It is unclear how one could interpret such solutions physically. We will encounter particular examples of the behavior of such a solution later in the paper.

3.2 The running coupling case

In technicolor theories, one typically has a technicolor non-abelian group, with a coupling $\alpha(p^2)$ that is renormalized, and an extended technicolor group that is broken at very high energies (on the order of the ETC scale), with an ETC coupling that is taken to be constant. We assume that the ETC coupling is below the critical value that would allow it to break chiral symmetry without the need of technicolor gauge interactions. Then, chiral symmetry will be broken at a scale where the running coupling $\alpha(p^2)$ gets strong enough, so that it, together with the ETC-coupling, can bring the system to criticality. For energies above that scale though, the gauge coupling is below its critical value. In that case, we expect the form of the solutions of the Schwinger-Dyson equations to change. First of all, we should insert the coupling λ inside the integral sign. Then, we can expect that, in a crude approximation, the relation between the coupling and the exponent of the function $G(t)$ will remain the same as for the constant coupling case.

In other words, we expect the quantity δ defined above to become purely imaginary,

so that the form of the solution for the fermion self-energy becomes

$$\Sigma(p^2) = \frac{\Sigma_0^2}{p} (p/\Sigma_0)^{2\tilde{\delta}}, \quad (19)$$

where $\tilde{\delta} = i\delta = \sqrt{1 - \frac{\pi\alpha(p^2)}{4\alpha_c}}$.

Such a naive analysis, however, neglects the complications arising from the fact that the coupling is running. There have been more careful analyses of the form of the self-energy and the way it evolves up to the ETC scale (see, for instance, [10]), and they have shown that, in general, the self energy at the ETC scale has approximately the form

$$\Sigma(\Lambda_{ETC}^2) \approx \frac{\Sigma_0^\omega}{\Lambda_{ETC}^{\omega-1}}, \quad (20)$$

where the power ω can be anywhere between 3 - the case of running coupling - and 2 - the limiting case of walking coupling, where we have a non-trivial UV fixed point in the theory. The parameter ω can also approach 1 in theories where the high-momentum enhancement is coming from 4-fermion interactions [10], which, as we have already seen, influence the boundary conditions of Eq.12.

The parameter ω is related in a very complicated way to δ , or, in other words, to the coupling λ . The reason for this complexity is the running of the gauge coupling, and the introduction of ETC 4-fermi couplings can make the situation even worse. The thing we can say here for sure is that we expect ω and δ to be negatively correlated, i.e. a larger coupling, over a large momentum region, should correspond to a smaller ω . This is intuitively reasonable, since usually Σ_0 is much smaller than Λ_{ETC} , and a larger coupling at large momenta should be able to produce larger self-energies $\Sigma_0(\Lambda_{ETC}^2)$. When studying the full non-linear Schwinger-Dyson equation, we also expect the parameter θ to be a function

of δ , which is nevertheless too complicated to be computed analytically. The relation in Eq.20 is going to be frequently used in the next section.

3.3 The quantization condition

The previous subsection dealt extensively with Eq.17 and its close connection to Eq.12. Eq.17 is a homogeneous Fredholm equation of the second kind. Unfortunately, as it stands, its kernel does not belong to L_2 , since the double integral

$$\int_0^\infty dt \int_0^\infty ds \frac{1}{(s+t)^2} \quad (21)$$

diverges. Therefore, it is not possible to apply the usual Fredholm theorems in this case [15]. An example of the singular behavior of Eq.17 is that, as we saw, the eigenvalues associated with it belong to a continuous spectrum of eigenvalues. The divergence of this double integral is logarithmic, and it comes from both the ultra-violet (UV) and infra-red (IR) regions. In both these cases, however, there are physical cut-offs that render the kernel square-integrable.

First of all, there is a UV-cut-off Λ associated with the new physics coming in at that scale. In technicolor models, for instance, the infinite upper bound of integrations of this kind is usually replaced by a finite cut-off Λ_{ETC} , where new physics in the form of extended-technicolor interactions come into play. Moreover, the role of the IR cut-off is played by the fermion self-energy $\Sigma(t)$ that should appear in the denominator of our kernel if we had not linearized our integral equation, i.e. if the kernel were of the more physical form $\frac{t}{(s+t)(t+\Sigma^2(t))}$. The linearization of our equation, as we shall see, while simplifying our analysis considerably, is not going to affect our final results in a qualitative way.

From the moment the kernel belongs to L_2 , we should expect Fredholm's theorems

to apply, and the spectrum of the eigenvalues of Eq.17 to become discrete. Let's see how this mechanism works in our case.

Taking care of the correct dimensionality of the quantities used, we can rewrite Eq. 17 as

$$\begin{aligned} c \left(\frac{s}{\Sigma_0^2} \right)^{\gamma-1} - c^* \left(\frac{s}{\Sigma_0^2} \right)^{-\gamma} &= \lambda \int_0^\infty dt \left[\frac{c \left(\frac{t}{\Sigma_0^2} \right)^{\gamma-1}}{t+s} - \frac{c^* \left(\frac{t}{\Sigma_0^2} \right)^{-\gamma}}{t+s} \right] = \\ &= \lambda \int_0^{\Lambda_{ETC}} dt \left[\frac{c \left(\frac{t}{\Sigma_0^2} \right)^{\gamma-1}}{t+s} - \frac{c^* \left(\frac{t}{\Sigma_0^2} \right)^{-\gamma}}{t+s} \right] + I(s), \end{aligned} \quad (22)$$

where $I(s) = \lambda \int_{\Lambda_{ETC}}^\infty dt \left[\frac{c \left(\frac{t}{\Sigma_0^2} \right)^{\gamma-1}}{t+s} - \frac{c^* \left(\frac{t}{\Sigma_0^2} \right)^{-\gamma}}{t+s} \right]$, and Σ_0 is the typical energy scale of the model, i.e. the value of the fermion self-energy at low energies, which is on the order of magnitude of the chiral symmetry breaking scale. We are now going to impose the condition $I(s) = 0$, and we are going to investigate what constrains such a condition imposes on the solutions.

By performing the above integrals, we have

$$\begin{aligned} I(s) = 0 &\Rightarrow \\ \Rightarrow &\frac{c \left(\frac{\Lambda_{ETC}}{\Sigma_0} \right)^{2(\gamma-1)}}{1-\gamma} F(1, 1-\gamma; 2-\gamma; -s/\Lambda_{ETC}^2) + \\ &+ \frac{c^* \left(\frac{\Lambda_{ETC}}{\Sigma_0} \right)^{-2\gamma}}{\gamma} F(1, \gamma; 1+\gamma; -s/\Lambda_{ETC}^2) = 0, \end{aligned} \quad (23)$$

where $F(a, b; c; z)$ is the usual hypergeometric function. Note that, had we included a momentum-independent self-energy term Σ_0^2 in the denominator of our kernel, in order

to formally maintain the kernel in L_2 , the only change would be in the argument z of the hypergeometric function, which would go from $-s/\Lambda_{ETC}^2$ to $-(s+\Sigma_0^2)/\Lambda_{ETC}^2$. It is therefore seen that the inclusion of such a term cannot alter our results substantially in the region of interest, which is $s \gg \Sigma_0^2$, and this is expected to remain true even if we insert in the denominator a more realistic momentum-dependent self-energy $\Sigma(t)$ ⁴.

The quantization condition that should derive from our equations should be of course momentum independent. In order to simplify our problem, we are going to make two different approximations that will enable us to derive such a condition in two different momentum regimes. First, we restrict the momentum regime to $\Sigma_0^2 \ll s \ll \Lambda_{ETC}^2$, which of course implies also that $\Sigma_0^2 \ll \Lambda_{ETC}^2$. Then, we can keep only the zeroth-order term of the series expansion of the hypergeometric function, and, setting $c = \|c\|e^{i2\delta\theta}$, with $0 < 2\delta\theta < 2\pi$, we have the momentum-independent equation

$$\frac{e^{i2\delta\theta}}{1/2 - i\delta} \left(\frac{\Lambda_{ETC}}{\Sigma_0} \right)^{i2\delta} + \frac{e^{-i2\delta\theta}}{1/2 + i\delta} \left(\frac{\Lambda_{ETC}}{\Sigma_0} \right)^{-i2\delta} = 0, \quad (24)$$

which is equivalent to the relation

$$2\delta\theta + \arctan(2\delta) + 2\delta \ln(\Lambda_{ETC}/\Sigma_0) = n\pi, \quad (25)$$

where n is an integer.

This quantization condition has been derived previously, using different techniques (see, for instance, Ref.[10]). If now, instead of taking the limit $s \ll \Lambda_{ETC}^2$, we take the limit $s \rightarrow \Lambda_{ETC}^2$, we get a similar quantization condition, but with the term $\arctan(2\delta)$ replaced by the constant $\pi/2$. This difference is considered to be an artifact of our derivation and the approximations involved in it. Moreover, it is not a significant change, and it is not

⁴This would lead us to the full, non-linear equation, where any discussion on eigenvalues and their spectrum is meaningless. We may apply our analysis, however, to the case of large momenta.

expected to alter the qualitative aspects of our results below, which rely mostly on the term $\ln(\Lambda_{ETC}/\Sigma_0)$. Since our quantization condition is going to be mostly used at energies $s \approx \Lambda_{ETC}^2$, we are going to use the form with the $\pi/2$ factor in it.

One could object that these results are not reliable, since in $I(s)$ we assume that the self-energies retain the same functional forms for $p > \Lambda_{ETC}$ as for momenta below Λ_{ETC} . However, a similar, more physical but more complicated analysis, using Heavyside functions which truncate the eigenfunctions at momenta above the cut-off, yields exactly the same results. Such an analysis was used recently on a completely different physical context [16], in order to derive a discrete eigenvalue spectrum out of an integral equation.

In the full non-linear theory, θ is in principle a function of δ . In our linearized equations, however, θ can be arbitrary. Since we have assumed that $\Sigma_0^2 \ll \Lambda_{ETC}^2$ and that $\theta > 0$, from Eq.25 we see that n has to be a positive integer larger than or equal to 1, i.e. $n \geq 1$. The relation appearing in Eq.25 provides us with a quantization condition, which is going to be central to the development of this paper. It has been previously derived using other methods, but solutions for $n > 1$ have not been really exploited.

Since, in the most general case, we are dealing with a non-abelian gauge group with a gauge coupling that is renormalized, the use of a constant gauge coupling as above is not very realistic. Therefore, in the discussion that follows, we are going to study the running coupling case.

As soon as we have to cope with a running gauge coupling, however, the situation changes dramatically. For the parameter δ is real at low momenta, but as we go to larger momenta it becomes imaginary. The problem is that the quantization condition that we derived previously is based on the assumption that δ is real (and constant). In addition, in

the running coupling case, δ is real in the momentum region where non-linearities become important, and the notion of the eigenvalue spectrum becomes problematic. The equation becomes so difficult for running δ that we were not able to compute analytically a quantization condition. Nevertheless, we know that, since the kernel of the linearized equation still belongs to L_2 ⁵, a quantization condition must exist, since the spectrum of eigenvalues must be discrete. To make a very crude approximation, we are going to assume that the same quantization condition as above is also valid for the running case, where in the place of δ , a real parameter δ is used in some “average” sense.

The use of such a parameter is based on the argument that, since there is chiral symmetry breaking and our equation possesses non-trivial solutions even in the running-coupling case, in a certain “average” sense we may consider δ to be real, i.e. the gauge coupling is above its critical value. This does not stop the actual δ parameter to reach imaginary values at large momenta. Actually, in the integral of Eq.24, the actual δ parameter is imaginary throughout the whole integration region. However, as we said previously, there are alternative ways for deriving exactly the same quantization condition, while staying inside the physical momentum region $0 \leq p \leq \Lambda_{ETC}$. The crudeness of such an analysis should not obscure the fact that the very nature of our equations, in the high-momentum, linear regime, makes them obey a certain quantization condition, even though it proves non-trivial, if not impossible, to find its exact analytical form. For instance, it is conceivable that, in the running coupling case, the ratio Λ_{ETC}/Σ_0 has a dependence on the integer n that is closer to a power law, instead of an exponential law, as implied by Eq.25. We do not try to analyse this, or other similar possibilities, here, as it would further complicate our analysis.

⁵we assume here that the gauge coupling does not possess a non-integrable singularity, but stays at finite values instead.

In the next section, we are going to analyse carefully the effects of the quantization condition, manipulated in the way described above, on the self-energy $\Sigma(p^2)$.

4 Physical interpretation

In this section, an attempt is made to find what physical consequences Eq.25 can have. In particular, we would like to see if the above quantization condition is in any way related to the appearance of the known fermions in three different generations. We cannot help remarking that the boundedness of operators similar to the one studied here is the source of quantization in ordinary quantum mechanics, like the energy levels of an electron confined in a finite box, or the energy levels of the hydrogen atom. In order to see if a mechanism of this sort could be qualitatively realistic in our case, we are going to neglect isospin mass splitting within the $SU(2)_L$ doublets, assuming that another mechanism is responsible for it, and we are going to consider only the up, charm and top for the quarks, and the electron, muon and tau for the leptons.

A very crude, order of magnitude inspection of their current masses reveals a hierarchy of a factor of about 200 among each subsequent generation, since, for the quarks, $m_u \approx 5$ MeV, $m_c \approx 1.5$ GeV, and we expect the top to have a mass of about $m_t \approx 170$ GeV. The top mass seems to be smaller than $200m_c$, but we should not forget that we are making an order-of-magnitude, qualitative discussion. This picture would correspond to linear trajectories on what is sometimes called the “Bjorken plot”. One could argue that, since the mass difference of the top and bottom quarks is so large, it is quite arbitrary to chose the upper partners of the quark doublets, and one could just as well consider the lower partners of the doublets instead, which would lead us to very different results. However, even though we do not have any rigorous argument towards that, we feel that, in a theory

that contains a minimum number of adjustable parameters, the top quark is the one that has the most “natural” mass, being the closest to the weak scale, where we believe that the fermion-mass origins lay. This leads us then to compare the top quark mass with the other two quarks having charge $+2/3$.

For the leptons, we have something similar happening, since $m_e \approx 0.5$ MeV, $m_\mu \approx 0.1$ GeV, and $m_\tau \approx 1.8$ GeV. We assume that the mechanism that makes m_τ considerably smaller than $200m_\mu$ is similar to the one making the expected value for m_t smaller than $200m_c$. Here, we neglect the upper partners of the lepton doublets, the neutrinos, leaving again to another mechanism the explanation for the smallness, or the vanishing, of their masses. In the following, we are going to assume that QCD or other effects can account for the quark-lepton mass difference, since the proposed mechanism cannot account for it.

Having this in mind for the ordinary fermions, it will be also useful to remind the reader that in technicolor theories, it is generally expected that Σ_0 , the maximum self-energy of the technifermions, is on the order of the chiral symmetry breaking scale $\Lambda_\chi \approx 1$ TeV. Moreover, since one expects the technifermions and the ordinary fermions to lie in the same representation of the extended technicolor group, before this breaks at scale Λ_{ETC} , the value of the technifermion self-energy at the extended technicolor scale Λ_{ETC} is expected to be on the order of the current mass of the corresponding ordinary fermion, i.e. $\Sigma_{Tf}(\Lambda_{ETC}^2) \approx m_f$. The above physical constrains are going to facilitate considerably the analysis of the quantization condition appearing in the previous section, and its possible connection to the fermion-generation puzzle.

Such a connection is inspired from the fact that the linear trajectories in the “Bjorken plot” could be attributed to some exponential dependence of the ratio of the two fundamental scales in the theory, Λ_{ETC}/Σ_0 , on a quantization integer index, as in Eq.25. The

fact that the top mass seems to be smaller than what expected for linear trajectories could be an indication that the dependence of the ratio Λ_{ETC}/Σ_0 on the quantization index does not follow an exponential law, but a power law or something similar instead, because of a possible modification of Eq.25 due to the fact that the gauge coupling is not constant. In the following, we do not try to modify the quantization condition, but describe an analysis that could be based, in principle, on other similar conditions.

It is essential to notice that the fact that the dependence of Λ_{ETC}/Σ_0 on the coupling δ , as in Eq.25, is non-analytic is a consequence of the non-perturbative nature of the Schwinger-Dyson approach that we chose to follow. Therefore, the results of any analysis based on this equation cannot be replicated by any perturbative considerations. Moreover, we should stress the fact that in what follows, we are going to refer to the ETC scales Λ_{ETC} in a very broad sense, and they should rather be viewed as new physical thresholds, since the discussion is not within the framework of conventional extended technicolor scenarios.

In Eq.25, three main physical parameters are involved: the extended technicolor scale Λ_{ETC} , the value of the fermion self-energy at zero momentum Σ_0 , and δ , which is related to the coupling λ . Another parameter which in the full, non-linear theory is a function of δ , the phase θ , is also entering the picture, and its value might have interesting consequences, as we will see later. Therefore, it seems as if our theory contains only two fundamental parameters, since the third can be determined by means of the quantization equation. One can further note that δ is mainly determined by the scale at which the former becomes strong, i.e. the confinement scale, and by the type of the non-abelian gauge group and the technifermion content of the theory. Therefore, if one assumes that the technicolor confinement scale is directly related to Σ_0 and the chiral symmetry breaking scale, or, in other words, the weak scale, one is essentially left with a single parameter, along with a

choice of the technicolor group and technifermion content, which renders this picture quite elegant. One should not forget, nevertheless, that our mechanism requires an additional parameter, which is the ETC effective 4-fermion coupling, which we take to be the same for all the fermions, and which influences, along with δ , the value of the power ω in Eq.20.

In the following, we consider δ as a free parameter that is not related to Λ_{TC} , since in our formalism δ is used in an “average” sense, and the connection between the two parameters seems non-trivial. *A priori*, we can fix a value for any two of these three parameters, and find a discrete set of values for the third one. Let us investigate all possible combinations. First, we can fix a value for δ and Λ_{ETC} , and find a discrete spectrum for Σ_0 . In this case, if we still want to pursue the argument of having to deal with essentially only one fundamental parameter in the theory, we have to assume that it is only the first member of the $\Sigma_0^{(n)}$ spectrum that is directly, and in a non-trivial way, associated with δ . $\Sigma_0^{(n)}$ is given by the expression

$$\Sigma_0^{(n)} = \Lambda_{ETC} e^{\theta + \frac{\pi}{4\delta}} e^{-n\pi/2\delta}. \quad (26)$$

We then assume that some of the solutions of this equation are related to the self-energies, at small momenta, of the technifermions corresponding to the three different generations of ordinary fermions. Then, inserting the above relation into Eq.20, we find

$$\Sigma^{(n)}(\Lambda_{ETC}^2) = \Lambda_{ETC} e^{\omega(\theta + \frac{\pi}{4\delta})} e^{-\frac{n\pi\omega}{2\delta}}. \quad (27)$$

Since we do not find any physical reason not to take consecutive solutions of the above equation, we consider the first three of them, for $n = 1, 2, 3$, and we take them to correspond to the technifermions associated with the top, charm, and up quarks respectively.

First of all, that would mean that we have to choose δ in such a way that $e^{-\frac{\pi\omega}{2\delta}} \approx 1/200$, which is the approximate hierarchy between consecutive fermion generations of characteristic mass scale $m_f^{(n)} \approx \Sigma^{(n)}(\Lambda_{ETC}^2)^6$. This would imply, with the use of Eq.27 and with a choice of a negligibly small phase θ , that the mass of the top quark is equal to $m_t \approx \Sigma^{(1)}(\Lambda_{ETC}^2) \approx \Lambda_{ETC}/14$. For $m_t \approx 170$ GeV, this gives an ETC scale $\Lambda_{ETC} \approx 2.4$ TeV. If we choose the values $\omega = 2$ and $\theta = 0$, this implies, from Eq.26, that $\Sigma_0^{(1)} \approx 640$ GeV, and $\delta \approx 0.59$. The choice of this value for ω has nothing in particular and is purely indicative. Ideally, one should be able to derive ω from δ and from the common ETC coupling of the fermions and technifermions.

Unfortunately, there are numerous phenomenological and theoretical problems with such a picture. First of all, it is not clear why we do not observe in nature lighter fermion generations associated with the solutions of the above equations for $n > 3$. Moreover, the solution for $n = 2$ or 3 implies the existence of technifermions having small self-energies at low momenta, which should make them observable in present experiments. However, we have not observed signs of their existence. This is a serious phenomenological draw-back of the mechanism described above. Furthermore, in this picture all the fermions are associated with the same ETC scale. A scale of about 2 TeV is unfortunately too low to adequately suppress flavor changing neutral currents in the light quark sector.

Another difficulty associated with this interpretation is the stability of such solutions. From effective-potential considerations (see Ref.[10], for example), it is clear that the effective potential is minimized for the maximum value of the self-energy. This makes stable only the solution for $n = 1$, and the solutions corresponding to higher n are unstable.

⁶ By n we index the fermion generations, and the spectrum of solutions deduced by the quantization condition.

This was the original reason for discarding solutions corresponding to higher n . We are now going to continue our discussion with some other possibilities that do not seem to possess these naturalness problems.

The next possibility we can think of is to fix the value of Σ_0 and δ , and find a quantization condition for the extended-technicolor scales. The relation resulting from that is

$$\Lambda_{ETC}^{(n)} = \Sigma_0 e^{-\theta - \frac{\pi}{4\delta}} e^{n\pi/2\delta}. \quad (28)$$

Inserting this expression into Eq.20, we get

$$\Sigma^{(n)}(\Lambda_{ETC}^2) = \Sigma_0 e^{(\omega-1)(\theta + \frac{\pi}{4\delta})} e^{-\frac{n\pi(\omega-1)}{2\delta}}. \quad (29)$$

If we want to reproduce the fermion hierarchy observed in nature, we must require that $e^{-\frac{\pi(\omega-1)}{2\delta}} \approx 1/200$. From Eq.29, setting $\omega = 2$ and $\theta = 0$, this would mean that $m_t \approx \Sigma_0/14$, so, for $m_t \approx 170$ GeV, this gives $\Sigma_0 \approx 2.4$ TeV.

In such a scenario, however, we have to be careful not to produce an unwanted hierarchy between the weak scale (or Σ_0) and the top quark mass (or $\Sigma^{(1)}(\Lambda_{ETC}^2)$), since our goal is to explain the maximum number of physical scales, using a minimum number of input parameters and mass hierarchies. In order to do that, we will have to use the phase θ , which up to now has not been really exploited. Taking θ to be close to $\pi/2$, and fixing $\omega = 2$, we find $\Sigma_0 \approx 500$ GeV. Quite interestingly, such a choice almost eliminates another hierarchy, the one between Σ_0 and $\Lambda_{ETC}^{(1)}$, suggesting that the ETC scale associated with the top quark is actually very close to 1 TeV. According to Eq.28, the ETC scales are then $\Lambda_{ETC}^{(1)} \approx 1.4$ TeV, $\Lambda_{ETC}^{(2)} \approx 290$ TeV, and $\Lambda_{ETC}^{(3)} \approx 58 \times 10^3$ TeV. Furthermore, we find $\delta \approx 0.3$. The ETC scale associated with the lightest generation, $\Lambda_{ETC}^{(3)}$, is much larger than the ones usually used in the literature, but we do not find any physical reason that would

prevent it from getting such a high value.

Moreover, we should caution the reader one more time that our results are purely indicative. Namely, we showed how a non-zero value of θ could fix various scales at reasonable values, but we should keep in mind that, in the full, non-linear equation, θ is determined by δ , and therefore it is not a free adjustable parameter. We may add as a speculation, that this dependence, which in a more careful study can be determined by the numerical solution of the integral equation, could be responsible for the fact that m_t is expected to be less than about $200m_c$. It should be noted that there are numerical indications that θ is non-zero [10]. This is an alternative way of getting around the problem of the deviation of the top quark mass from the linear “Bjorken trajectories”, other than the one in which we modify the form of the quantization condition.

A remark along the same lines can be written about the power ω ; the fact that we took $\omega = 2$ and $\delta \approx 0.59$ in our previous example, should have made us choose a larger value for ω in this example, instead of using $\omega = 2$ again, since in the present example δ is smaller, i.e. $\delta \approx 0.3$. A larger ω would also bring $\Lambda_{ETC}^{(3)}$ down to a smaller, more acceptable value. However, since we do not know the exact dependence of ω on δ , we do not feel that we should further complicate the picture with changes in parameters that do not add anything crucial to the qualitative behavior of the mechanism.

This second way of looking at the quantization condition does not seem to have the stability problem that the previous solutions had, nor does it predict any new particles at low scales. Furthermore, it is conceptually very close to the idea proposed [7] and used [17] recently, according to which each fermion is associated with different extended-technicolor (ETC) scales, instead of having a single technicolor scale for all of them, as more conventional technicolor models suggest. The difference here, of course, is that the

hierarchy of ETC scales is not introduced arbitrarily, but is produced by a specific underlying mechanism. Such a mechanism would bring the chiral symmetry breaking scale in the picture naturally, as associated with the (common) technifermion self-energy at low momenta. Then, it would automatically associate the lighter generations to the higher extended-technicolor scales.

We also see that it is not necessary to assign a different ETC coupling to each fermion any more, as conventional technicolor models do, since this burdens the model with too many parameters. The change of the ETC-scales is enough to account for the change of the fermion masses from generation to generation. In addition, one can argue that we only have three generations, or equivalently that the solutions for $n > 3$ do not make sense, because there are some new physics, above the scale $\Lambda_{ETC}^{(3)}$, making our analysis not applicable any more. This would give more predictive power to the proposed mechanism, since by using the known fermion spectrum we could have a feel of the order of magnitude where new physics, beyond extended technicolor, enter into the picture. Note that such large energy scales ($\gtrsim 10^5$ TeV) have been observed in highly-energetic cosmic rays [18]. These energies are still very far below a possible grand-unification scale or the Planck scale.

A great advantage of such a mechanism is that it can avoid large flavor-changing neutral currents, since the ETC scale associated with the light quarks is high, while at the same time it can generate large bottom and top quark masses, since they are associated with a much smaller ETC scale. Of course, this also implies the existence of large FCNC associated with the third fermion generation, as well as non-negligible corrections to the $Z^0 \rightarrow \bar{b}b$ vertex, effects that should be detectable in precision experiments. The smallness of the ETC scale associated with the top quark has been shown to serve two more phenomenological purposes: it can keep small not only the S parameter, since a “walking” mechanism

requiring many technifermions is no longer needed to generate large heavy quark masses, but also the $\Delta\rho$ parameter [17].

The problem with this mechanism is that it is theoretically unclear how each fermion generation is associated with each scale. Unlike the usual tumbling mechanism, where the scales introduced are the energies where the gauge interactions become so strong that they break the gauge group to a smaller one, the mechanism proposed here does not possess, at first sight at least, such a straightforward interpretation. It would seem that it is only for specific ETC scales that the Schwinger-Dyson equation can have non-trivial solutions and break chiral symmetry. Then, it is this very dynamical symmetry breaking that causes the ETC group to break successively at these scales down to smaller groups, reproducing a mechanism similar to “tumbling”. The correct physical interpretation of this phenomenon is a very challenging model-building problem that we can hardly address here, and we will return to it, along with a more general physical discussion, in the next section.

We next go to the last remaining possibility, which is to fix Λ_{ETC} and Σ_0 , and to derive a quantization condition for δ . The physical interpretation for such a picture could be more straightforward than the previous one, since it would signal that we have all the technifermion self-energies starting-off at low momenta from their common initial value Σ_0 , and then drop up to their common ETC scale according to different anomalous dimensions, i.e. with different couplings. This could be very interesting from the point of view of model-building, since such a behavior could be attributed to having technifermions sitting in different representations of the same technicolor group, or having them interact with different technicolor groups altogether. Unfortunately, since the relation between ω and δ is non-trivial, especially in theories where one employs 4-fermion-induced high-momentum enhancement, we do not pursue this analysis further, but merely contend ourselves to

stating this interesting possibility.

5 Conclusions

In this work, we have attempted to construct a mechanism that would explain the mass hierarchy of the three fermion generations, in a context of dynamical electroweak and chiral symmetry breaking models. We have tried to achieve this by using a minimum number of input parameters, which makes these models more natural.

The explanation of the mass hierarchies in Nature is however a highly non-trivial problem, and attempts to solve it usually give rise to serious complications. In our case, the solution that is both solvable, after using several approximations, and phenomenologically acceptable, is the one in which we fix the weak scale, which is closely related to Σ_0 , and then the ETC scales follow from a quantization condition. Unfortunately, such an interpretation is not along the lines of conventional wisdom in present-day particle physics. Let us see why this is so.

At first, we have to understand what the three fermion generations correspond to in this picture. They appear as the same reality that replicates itself, and manifests itself into three different ways. In the everyday world, we can observe only the lowest-energy manifestations of that reality, by means of the lightest fermion generation. It is only when we go to higher energies that we can see its higher-energy manifestations. The role of ETC scales, however, and the exact way in which they enter in this process, is still unclear. They appear as the scales which can lead, when chiral symmetry breaking sets in, to a technifermion self-energy that is very close to the weak scale at low momenta.

Moreover, what is usually expected in model building is for high-energy physics to

“feed-down” their effects to lower energies. The picture as presented here, on the contrary, seems to do the opposite; we first fix the weak scale and the coupling, and then we find the corresponding spectrum of ETC scales. It is as if lower-energy physics determine the behavior of higher-energy physics.

This, however, is not a completely new phenomenon in the physical world. As a very naive and simplistic example, we take the harmonic oscillator. One can completely define this quantum-mechanical system by specifying its fundamental frequency ω_0 . After solving the equations, however, we predict a whole spectrum of frequencies that are arbitrarily larger than the fundamental one, in a similar way that our equations predict a spectrum of ETC scales much higher than the weak scale. Moreover, the reason for the appearance of a discrete spectrum in quantum mechanics is not always the existence of a bound state of two particles, but can also be the confinement of a particle in a finite space region.

We are very much aware of the fact that analogies like the one above can lead to serious misconceptions. For instance, in our case the ETC scales are supposed to be physical cut-offs, and not the energy levels of a system of particles. The message that we want to convey should be clear nevertheless: we want to consider the weak scale, or equivalently the scale where new, strongly-interacting physics come into the picture at around 1 TeV, as a fundamental physical parameter which, by its inverse, sets a certain spatial scale. Within that finite space, the behavior of the Schwinger-Dyson equations generate a discrete spectrum of energy scales (cut-offs), which could possibly be identified with the ETC scales of technicolor theories.

In addition, we should not forget that the fermion masses are much closer to the weak scale, rather than the Planck scale, so it does not seem to us too unnatural trying to explain them in terms of physics coming in at the weak scale, rather than expecting

“Planck-” or higher-scale physics to “feed down” their effect directly to the fermion masses. Of course, the weak scale itself could still be determined by some unknown high-energy physics, appearing at the “Planck”, or even at the highest ETC, scale. Therefore, the proposed mechanism, when seen from this point of view, does not completely violate the way high-energy physics determine low-energy physics. It just gives the weak scale a more active and direct role in the fermion mass generation, while leaving for the Planck-scale, or for any other scale that determines the weak scale, only an indirect role.

Such a picture still gives a very limited explanation of the mass hierarchies observed in nature. The huge hierarchy between the Planck scale and the weak scale still remains a mystery. Furthermore, the QCD scale is another scale that is not accounted for in this picture. Even though it is conceivable that these scales can be explained by a similar paradigm, trying to incorporate them in the present discussion would be over-ambitious.

To conclude, we would like to add the following comments. The physical interpretation of the proposed picture may still seem elusive. In such a case, it would still be interesting, as well as useful, to consider the formulas given here as purely phenomenological, that merely describe, which they seem to do indeed, instead of explaining, the true situation. We should then await for a better understanding of the whole process. Within the same framework, it would be also very useful to perform a more detailed and careful mathematical analysis of the quantization condition, and a more thorough investigation of possible models and physical processes that could explain the inner works of this mechanism.

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References

- [1] H. Burkhardt and J. Steinberger in J. D. Jackson (ed.) *et al*: Annual review of nuclear and particle science, vol. **41**, 55 (1991)
- [2] See, for instance, D. V. Nanopoulos and D. A. Ross, Nucl. Phys. **B157**, 273 (1979); and Phys.Lett. **B118**, 99 (1982)
- [3] For a relatively recent study, see H. Schlereth, Phys. Lett. **B243**, 128 (1990)
- [4] See for example P. Candelas, C. A. Luetken and R. Schimmrigk, Nucl. Phys. **B309**, 113 (1988)
- [5] J. Chakrabarti, M. Popovic and R. N. Mohapatra, Phys. Rev. **D21**, 3212 (1980) and references therein; J. L. Alonso, V. Azcoiti and A. Cruz, Phys. Rev. **D26**, 691 (1982); for more recent studies, see D. B. Kaplan and M. Schmaltz, UCSD Preprint UCSD/PTH 93-30, October 1993, and references therein. This list is only indicative, and by no means should it be considered as exhaustive.
- [6] S. Dimopoulos, L. J. Hall and S. Raby, Phys. Rev. Lett. **68**, 1984 (1992); Phys. Rev. **D45**, 4192 (1992)
- [7] M. B. Einhorn and D. Nash, Nucl. Phys. **B371**, 32 (1992)
- [8] For a classic review, see E. Farhi and L. Susskind, Phys. Rep. **74**, 277 (1981)
- [9] T. Appelquist and J. Terning, Yale Preprint YCTP-P21-93, November 1993
- [10] See, for instance, T. Takeuchi, Phys. Rev. **D40**, 2697 (1989)

- [11] B. Holdom, Phys. Lett. **B150**, 301 (1985); T. Appelquist and L.C.R. Wijewardhana, Phys. Rev. **D36**, 568 (1987); T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. **D35**, 774 (1987); T. Appelquist, M. Einhorn, T. Takeuchi, and L. C. R. Wijewardhana, Phys. Lett. **B220**, 223 (1989); V. A. Miransky and K. Yamawaki, Mod. Phys. Lett. **A4** (1989) 129; K. Matumoto, Prog. Theor. Phys. Lett. **81** (1989), 277; V. A. Miransky, M. Tanabashi, and K. Yamawaki, Phys. Lett. **B221** (1989), 177; V. A. Miransky, M. Tanabashi, and K. Yamawaki, Mod. Phys. Lett. **A4** (1989), 1043.
- [12] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10**, 2428 (1979)
- [13] Y. Nambu and Jona-Lasinio, in C. H. Lai (Ed.): Gauge theory of weak and electromagnetic interactions, 104 (1982)
- [14] R. Holdom, Phys. Rev. Lett. **62**, 997 (1989)
- [15] See, for instance, F. G. Tricomi: Integral Equations. Interscience Publishers, Inc., New York (1957), p. 65
- [16] J. C. Collins and P. V. Landshoff, Phys. Lett. **B276**, 196 (1992)
- [17] T. Appelquist and G. Triantaphyllou, Phys. Lett. **B278**, 345 (1992)
- [18] D. J. Bird *et al.*, Phys. Rev. Lett. **71**, 3401 (1993)